# On Page's examples challenging the entropy bound

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According to the entropy bound, the entropy of a complete physical system can be universally bounded in terms of its circumscribing radius and total gravitating energy. Page's three recent candidates for counterexamples to the bound are here clarified and refuted by stressing that the energies of all essential parts of the system must be included in the energy the bound speaks about. Additionally, in response to an oft heard claim revived by Page, I give a short argument showing why the entropy bound is obeyed at low temperatures by a complete system. Finally, I remark that Page's renewed appeal to the venerable "many species" argument against the entropy bound seems to be inconsistent with quantum field theory.

## I. INTRODUCTION

In 1981 I proposed [1] that the entropy of a complete physical system in three spatial dimensions whose total mass-energy is $E$, and which fits in a sphere of radius $R$, is necessarily bounded from above,

$$S \leq 2\pi ER/\hbar c, \tag{1}$$

(henceforth I set $\hbar = c = 1$). The motivation comes from *gedanken* experiments in which an entropy bearing object is deposited in a black hole with a minimum of energy; a violation of the generalized second law seems to occur unless the bound applies to the object [1–3]. A variant *gedanken* experiment [4] in which the object is freely dropped into the hole again suggests that an entropy bound of the above form must be valid. A number of direct calculations of entropy of systems with no black hole involved have supported the validity of the bound for systems with negligible self-gravity [2,5,6]. The entropy bound (1) is related to the holographic bound [4,7], and is often a stronger restriction on the entropy. Because of the intimate relation of information to entropy, bound (1) implies a fundamental restriction on the capacity of information storage and communication systems [5,6,8,9]. This may become practically important in the distant foreseeable future (the same cannot be said of the vastly more lenient holographic information bound).

Over the years Page has proposed a number of counterexamples to the entropy bound [10–13]. In ref. [14] I refuted the two from ref. [10], while in ref. [15] I showed Page's proposed substitute [10] for bound (1) to be violated. In the present paper I concentrate on Page's

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recent candidates for counterexamples, and show that they do not constitute violations of the entropy bound as originally stated. It is common for critics of bound (1) to fail to include in $E$ the energy of some part of the system, contrary to the stipulations [1,2,5,14] that the bound applies only to a complete system—one that could be dropped whole into a black hole—and that $E$ in the bare inequality (1) means the system's gravitating energy (gravitating energy, of course, is immune to the well known arbitrariness of the zero of energy). Excluding part of the energy from $E$ empties the bound of meaning. Thus, an harmonic oscillator with sufficiently large mass (which makes the energy spacing of its levels small) manifestly violates the bound unless one includes the oscillator's mass in $E$ [5]. Yet in all his new examples, Page drops from $E$ the energy of some part of the system.

Page's first example [11,12] is a self-interacting scalar field with a symmetric double well potential, with the field vanishing at a certain spherical boundary of radius $R$. Page correctly points out that quantum tunnelling between the wells splits the classically degenerate ground states into a new ground state and an excited state a very small energy $\Delta E$ above the ground state. Page hastily concludes that this violates bound (1) because the entropy of a mixed state built out of the two can reach $\ln 2$, while $2\pi R \Delta E$ can be very small in natural units ($\hbar = c = 1$). As I show in Sec. II, Page's exclusion of the classical energy of the configuration (much larger than $\Delta E$) is unjustified because the last contributes to the field's gravitating energy. When $E$ includes it and the indispensable minimum wall energy, bound (1) is always obeyed.

In all of his new examples Page excludes contributions to $E$ from the passive parts of the system. He quotes extensively from Schiffer's and my paper [16] as license for this procedure. The context is this. In the 1980's, based on the many cases studied numerically, we had toyed with the idea that a strong form of bound (1) might apply, whereby $S$ and $E$ in inequality (1) refer solely to a collection of *noninteracting* massless quanta, with the contributions to $E$ from the cavity containing them ignored. We conjectured a theorem to this effect, which we proved for massless scalar quanta confined to a cavity of arbitrary topology by Dirichlet boundary conditions [6,16]. We also sketched the proof of a generalization of that theorem for photons or massless neutrinos; we considered only the wave equation, but did not go into details about boundary conditions or constraint equations which are very specific for electromagnetism or fermion fields. Despite some efforts, we never completed these proofs. So in the intervening decade I have used bound (1) in its original formulation [2,4,7]. Page's new examples, which hark back to a version of the entropy bound which did not become established generically, teach us little about the true bound on entropy, and certainly do not constitute counterexamples to it.

In his second example [11], an onion-like arrangement of a number of infinitely conducting concentric partitions separating electromagnetic fields in different states of excitation, Page establishes that some mixed states of the electromagnetic field disobey the strong bound $S < 2\pi(E - E_{\text{vac}})R$, where $E - E_{\text{vac}}$ refers only to the electromagnetic energy above the vacuum, and $R$ to the radius of the outermost partition. But this is *not* a violation of Schiffer-Bekenstein theorem [6,16]; the theorem (strictly proved only for scalar quanta) only claims that the 'strong entropy bound' is obeyed by a *noninteracting* field in a cavity, whereas in Page's onion the electromagnetic field *interacts* throughout the volume of the sphere of radius $R$ with certain massive charged fields describing the conducting material of the partitions. My semiclassical discussion of the conducting partitions in Sec. III shows

that even without specification of the nature of the charge carriers, the complete system does obey the original entropy bound (1).

Page's third example [12] is a closed loop of coaxial cable of total length $L$ coiled up inside radius $R$, with $R \ll L$, which confines an electromagnetic field. Page gives hand waving arguments that the lowest travelling frequencies are $2\pi/L$. Citing again the electromagnetic version of the Schiffer-Bekenstein theorem for license, Page estimates $E$ for the cable containing a mixed state based on the two lowest lying modes to be of order $1/L$. Since the state can have entropy of order unity, Page argues that the entropy bound is violated since $RE \ll 1$. If Page is correct about the low frequencies, then the Schiffer-Bekenstein theorem cannot be proved for electromagnetic fields confined to a waveguide with a not simply connected crossection. At any rate, the complete system includes the charged carriers which form the cable, and just as in the previous example, these contribute enough to $E$ to make bound (1) work (Sec. IV).

Page revives [11] Deutsch's old objection to the entropy bound to the effect that weak thermal excitation of the system involves a violation [17]. I give in Sec. V a brief argument showing why the entropy bound is always obeyed by a *complete* system, even at low temperature. The problem of entropy bound violations at low temperatures is evidently a red herring ! The above argument also illustrates how to deal with any mixed state which ascribes low probabilities to the high energy pure states.

Page also revives the old argument [18–20] that bound (1) will be violated when there is a virtually unlimited number of particle species in nature. I have earlier refuted this [2,3,14,15]. In addition, as I discuss in Sec. VI, one knows today that the vacuum of quantum field theory is gravitationally unstable when many species of particles exist [21]. Such instability makes Page's second revived argument moot.

## II. NONLINEAR SCALAR FIELD

Page's first example [11,12] deals with a self-interacting scalar field with a multiwell potential, and considers configurations of the field which vanish at a certain boundary of radius $R$. For the symmetric double well potential Page correctly points out that the classically degenerate ground states of the field, each localized in one well, engender, by quantum tunnelling between the wells, a new ground state $\psi_0$ (energy $E_0$) with equal amplitude at each well and a first excited state $\psi_1$ a very small energy $E_1 - E_0$ above the ground state. He then hastily concludes that this violates the bound because the entropy of a mixed state built out of $\psi_0$ and $\psi_1$ can reach $\ln 2$ (ground and excited states equally probable), while $(E_1 - E_0)R$ can be very small in natural units ($\hbar = c = 1$). In this interpretation Page considers the energy $E$ mentioned in the bound as the energy measured *above* the ground state. It would indeed be so if the ground state referred to a spatially unrestricted configuration, because then the bottom of the potential well would be the correct zero of energy (neglecting zero point fluctuations).

But since the field is required to vanish at radius $R$, the energy $E_0$ of the described ground state is a function of $R$, and it makes little sense to take it as the zero of energy. For example, by expanding the system can do work ($-\partial E/\partial R \neq 0$), so that its gravitating energy changes, and cannot be taken as zero for all $R$. The gravitating energy for the equally likely mixture of ground and excited states should be identified with $\frac{1}{2}(E_0(R) + E_1(R)) \approx E_0(R)$, which

does go to zero as $R \to \infty$, and is, therefore, properly calibrated with respect to the global ground states of the theory.. The exponential smallness of $E_1 - E_0$, which Page pounced upon, is not very relevant, as will transpire.

Since there are no solitons in $D = 1 + 3$ spacetime, a finite sized field configuration [the only interesting case—see (1)] must be confined by a "wall". There are three parts to the energy $E$ of the total system (before tunnelling is taken into account): the classical energy $E_c$ of the field configuration concentrated around one well but vanishing at radial coordinate $r = R$, the quantum correction $E_v$ due to the zero point fluctuations about the classical configuration, and $E_w$, the energy of the "wall" at $r = R$. As I show below, $E_w$ is at least of the same order as $E_c$, and both dominate Page's energy.

## A. Classical two-well configurations

The double well potential field theory comes from the lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}\lambda(\phi^2 - \phi_m^2)^2. \tag{2}$$

This gives the field equation

$$\partial_\mu\,\partial^\mu\phi - \lambda\phi(\phi^2 - \phi_m^2) = 0. \tag{3}$$

Every spherically symmetric configuration inside a spherical box of radius $R$ will thus satisfy (I use standard spherical coordinates; $'$ denotes derivative w.r.t. to $r$)

$$r^{-2}(r^2\phi')' - \lambda\phi(\phi^2 - \phi_m^2) = 0. \tag{4}$$

Regularity requires that $\phi' = 0$ at $r = 0$. Page chooses $\phi = 0$ at $r = R$. The energy of such a configuration will be

$$E_c = \frac{1}{2}\int_0^R \left[\phi'^2 + \frac{1}{2}\lambda(\phi^2 - \phi_m^2)^2\right]r^2\,dr. \tag{5}$$

Since one is interested in the ground state, I require that $\phi$ have its first zero at $r = R$. Multiplying Eq. (4) by $r^2\phi$ and integrating over the box allows, after integration by parts and use of the boundary conditions, to show that

$$\int_0^R \phi'^2 r^2\,dr = \lambda\int_0^R (\phi_m^2 - \phi^2)\phi^2 r^2\,dr \tag{6}$$

whereby

$$E_c = \frac{1}{4}\lambda\int_0^R (\phi_m^4 - \phi^4)r^2\,dr. \tag{7}$$

It proves convenient to adopt a new, dimensionless, coordinate $x \equiv \sqrt{\lambda}\phi_m\,r$ and a dimensionless scalar $\Phi \equiv \phi/\phi_m$. Then Eq. (4) turns into a parameter-less equation:

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\Phi}{dx}\right) + \Phi(1 - \Phi^2) = 0. \tag{8}$$

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Using $d\Phi/dx = 0$ at $x = 0$ one may integrate the equation to get

$$\frac{d\Phi}{dx} = -\frac{1}{x^2} \int_0^x \Phi(1 - \Phi^2)\, x^2\, dx. \tag{9}$$

If the integration starts with $\Phi(0) > 1$, then by continuity the r.h.s. here is positive for small $x$, so that $\Phi$ grows. There is thus no way for the r.h.s. to switch sign, so $\Phi(x)$ is monotonically increasing and can never have a zero. If $\Phi(0) = 1$, it is obvious that the solution of Eq. (9) is $\Phi(x) \equiv 1$ which cannot satisfy the boundary condition at $r = R$. Thus the classical ground state configuration we are after requires $\Phi(0) < 1$.

When $\Phi(0) < 1$ it can also be seen from Eq. (9) that $\Phi$ is monotonically decreasing with $x$. For a particular $\Phi(0)$, $\Phi(x)$ will reach its first zero at a particular $x$ which I refer to as $x_0$. This can serve as the parameter singling out the solution in lieu of $\Phi(0)$. One thus has a family of ground state configurations $\Phi(x, x_0)$. Each such configuration corresponds to a box of radius $R = x_0(\sqrt{\lambda}\phi_m)^{-1}$. In terms of the new variables one can write Eq. (7) as

$$E_c = \frac{x_0}{4\lambda R} \int_0^{x_0} (1 - \Phi^4) x^2\, dx. \tag{10}$$

The dependence $E_c \propto \lambda^{-1}$ is well known from kink solutions of (3) in $D = 1+1$ [22], where the role of $R^{-1}$ is played by the effective mass of the field. Numerical integration of Eq. (8) shows that the factor $x_0 \int_0^{x_0}(1-\Phi^4)x^2\, dx$ grows monotonically from 32.47 for $\Phi(0) = 0$ ($x_0 = 3.1416$) to 232.23 for $\Phi(0) = 0.98$ ($x_0 = 5.45$) to infinity as $\Phi(0) \to 1$ ($x_0 \to \infty$). Since $E_c$ is not exponentially small, the quantum tunnelling corrections that Page discussed are negligible, so one need only add to $E_c$ the zero point fluctuations energy $E_v$ plus the wall energy $E_w$ to get the full energy of the ground state, $E_0$. This plays the role of $E$ in the bound (1).

I shall not bother to calculate $E_v$. This can be done by present techniques only for the weak coupling case $\lambda < 1$ [22]. It is then found in other circumstances, e.g. the $D = 1 + 1$ kink, that $E_v$ is small compared to $E_c$. The situation for large $\lambda$ (the strong coupling regime) is unclear. However, it is appropriate to recall here that the theory (2) is trivial in that it makes true mathematical sense only in the case $\lambda = 0$ [23]. Theorists use it for $\lambda \neq 0$ to obtain insights which are probably trustworthy in the small $\lambda$ regime, but probably not for large $\lambda$.

I now set a lower bound on $E_w$. A look at Eq. (10) shows that for $\Phi(0) \ll 1$ and so $\Phi(x) \ll 1$), $E_c$ scales as $x_0^4/R \propto R^3$. Numerically the exponent here only drops a little as $\Phi(0)$ increases; for example, it is 2.86 for $\Phi(0) = 0.98$. So I take it as 3. On virtual work grounds (consider expanding $R$ a little bit) the $R^3$ dependence means the $\phi$ field exerts a suction (negative pressure) of dimension $\approx (3E_c/4\pi R^3)$ on the inner side of the wall. By examining the force balance on a small cap of the wall, one sees that in order for the wall to withstand the negative pressure, it must support a compression (force per unit length) $\tau \approx (3E_c/8\pi R^2)$ [14]. Under this compression vibrations on the wall will propagate superluminally unless the surface energy density is at least as big as $\tau$ (dominant energy condition). Thus one may conclude that the wall (area $4\pi R^2$) must have (positive) energy $E_w > 3E_c/2$ which *adds* to $E_c$ to give $E > 5E_c/2$.

For $\Phi(0)$ very close to unity the coefficient is somewhat lower than 5/2. However, by then $E_cR$ is already much larger than the corresponding quantity for $\Phi(0) \ll 1$ (six times larger for $\Phi(0) = 0.98$). Using the value of $E_c$ for $\Phi(0) \ll 1$ from the preceding argument,

I thus conclude that for all physically relevant $\Phi(0)$, $2\pi ER > 127.5\lambda^{-1}$. This is certainly not exponentially small with $R$ as Page claimed ! True, formally it seems possible to have a violation of the bound on the entropy $\ln 2$ of the 50% mixture of ground and excited states whenever $\lambda > 127.5/\ln 2 = 183.95$. However, this is the strong coupling regime. For all one knows the zero point energy may then become important and tip the scales in favor of the entropy bound. At any rate, because the theory (2) is trivial, one is more likely to be overstepping here the bounds of its applicability than to be witnessing a violation of the entropy bound at large $\lambda$.

In summary, we have found that whenever the calculation is meaningful ($\lambda$ not large), the entropy bound (1) is satisfied provided $E$ includes all contributions to the energy. In his later defense against this observation, Page [12] cites my paper with Schiffer [16] as an excuse for including in $E$ just the excitation energy above the classical ground state. However, he neglects to point out that we ourselves restricted use of this approach to an assembly of quanta of a massless *noninteracting* field.

I should also mention that in $D = 1 + 1$ spacetime it is possible to find analytically all static classical configurations (and their energies) for the theory (2) in a box, and the distribution of energy levels is such that the entropy bound is sustained [24].


## B. Multiwell potential

Page also confronts bound (1) with a theory like (2) but with a potential having three equivalent wells. Pressumably one would like one of these centered at $\phi = 0$, with the other two flanking it symmetrically. Then Page's conclusion that there are three exponentially close states (in energy) is untenable. This would require three classically degenerate configurations, which certainly exist in open space (field $\phi$ fixed at one of three well bottoms). However, one is here considering a finite region of radius $R$ with $\phi = 0$ on the boundary. One exact solution is indeed $\phi = 0$, and it has zero energy (the zero point fluctuation energy correction will, however, depend on $R$). Then there are two degenerate solutions in which the field starts at $r = 0$ in one side well and then moves to the central one with $\phi \to 0$ as $r \to R$. By analogy with our earlier calculations, the common classical energy of these two configurations will be of $O\left(8(\lambda R)^{-1}\right)$. It cannot thus be regarded as the zero of energy; this role falls to the energy of the $\phi = 0$ configuration. When tunnelling between wells is taken into account, one has a truly unique ground state and two excited states of classical origin split slightly in energy (plus the usual gamut of quantum excitations). The entropy of an equally weighted mixture of these states is $\ln 3$. The mean energy $E$ is $\frac{2}{3}(E_c + E_w)$ of an excited state, that is $E = O(10(\lambda R)^{-1})$, so the entropy bound is satisfied, at least in the weak coupling regime where the theory makes sense.

When the potential has $n = 5, 7, 9, \cdots$ equivalent wells with one centered at $\phi = 0$ and the rest disposed symmetrically about it, there will be a single zero-energy configuration ($\phi \equiv 0$), and $\frac{1}{2}(n - 1)$ pairs of degenerate configurations with succesively ascending $R$-dependent energies. For $n = 4, 6, 8, \cdots$ wells there is no zero-energy configuration, but there are $\frac{1}{2}n$ pairs of degenerate configurations with $R$ dependent energies. Because of the extra energy splitting appearing here already classically, I expect, by analogy with the previous results, that the appropriate mean configuration energy (perhaps supplemented by wall energy), when multiplied by $2\pi R$, will bound the $\ln n$ entropy from above.

### III. ELECTROMAGNETIC ONION

As a second counterexample to the entropy bound, Page proposes [11] a sphere of radius $R$ partitioned into $n$ concentric shells; the partitions and the inner and outer boundaries are regarded as infinitely conducting. He points out that the lowest ($\ell = 1$) three magnetic-type electromagnetic modes in the shell of median radius $r$ have frequency $\omega \approx 1/r$. Since there are $3n$ such modes (three for each shell), Page imagines populating now one, then another and so on with a single photon of energy $\sim 1/r$ for the appropriate $r$. These one-photon states allow him to form a density matrix which, for equally weighted states, gives entropy $\ln(3n)$ and mean energy $\sim 2/R$ (since $R/2$ is the median radius of the shells if they are uniformly thick). Page concludes that bound (1) is violated because the entropy grows with $n$ but the mean energy does not.

Page has again missed out part of the energy. The modes he needs owe their existence to the infinitely conducting partitions that confine them, each to its own shell. Were passage between shells possible, then old work [1] already established that the entropy bound works for the electromagnetic field confined to an empty sphere (or for that matter confined to any parallelepiped [5]. To be highly conducting, the envisaged partitions must contain a certain number of charge carriers, whose aggregated masses turn out to contribute enough to the system's total energy $E$ to make it as large as required by the entropy bound (1). Ignoring the masses of the charge carriers goes against the condition that the bound applies to a complete system: the carriers are an essential component, so their gravitating energy has to included in $E$.

I assume all partitions to have equal thickness $d$. One mechanism that can block the waves from crossing a partition is a high plasma frequency $\omega_p$ of the charge carriers in the partitions. We know [25] that in a plasma model of a conductor with collisionless charge carriers, the electromagnetic wave vector for frequency $\omega$ is $k = \omega(1 - \omega_p^2/\omega^2)^{1/2}$, so that if $\omega < \omega_p$, the fields do not propagate as a wave. Nevertheless they do penetrate a distance $\delta = \omega^{-1}(\omega_p^2/\omega^2 - 1)^{-1/2} > \omega_p^{-1}$ into the plasma before their amplitudes become insignificant. In order to prevent these evanescent waves from bridging a partition, one must thus require $\delta < d$, i.e., $\omega_p d > 1$. But

$$\omega_p{}^2 = 4\pi \mathcal{N} e^2/m \tag{11}$$

where $\mathcal{N}$ is the density of charge carriers of charge $e$ and mass $m$. Since $d < R/n$, all this gives us $(4\pi R^2 d)\mathcal{N} > mn^2 d/e^2$. Now $4\pi R^2 d$ is the volume of material in the outermost partition. Properly accounting for the variation of partition area with its order $i$ in the sequence (we employ the sum $\sum i^2$), tells us that for $n \gg 1$ the total mass-energy in charge carriers in all the partitions is $E \approx nm(4\pi R^2 d/3)\mathcal{N}$. Substituting our previous bound on $(4\pi R^2 d)\mathcal{N}$, I get $E > \frac{1}{3}n^3 m^2 d/e^2$.

Now a charge carrier's Compton length has to be smaller than $d$, for otherwise the carriers would not be confined to the partitions; hence $md > 1$. As a matter of principle $e^2 < 1$, because more strongly coupled electrodynamics makes structures, e.g. atoms and partitions, which are held together electrically, unstable [26] (in our world $e^2 < 10^{-2}$). Therefore, since $d < R/n$ one gets $E > \frac{1}{3}n^4 R^{-1}$, which strongly dominates $\sim 2/R$, the energy in photons that Page gets. In particular $2\pi RE > 2n^4$, which is always much larger than the entropy in photons $\ln(3n)$.

The only alternative mechanism for keeping electromagnetic waves from penetrating into a conductor is the skin effect [25]. The skin depth is $\delta \approx (2\pi\omega\sigma)^{-1/2}$, where $\sigma$ is the conductivity. In the simple Drude model [25], $\sigma = \mathcal{N}e^2(m/\tau - \imath m\omega)^{-1}$, where $\tau$ is the slowing-down timescale for a charge carrier due to collisions, and $\imath = \sqrt{-1}$. The formula for $\delta$ refers to an Ohmic (real) conductivity rather than an inductive (imaginary) one. Thus one must demand that $\omega \ll 1/\tau$. But then

$$\delta \gg (2\pi\mathcal{N}e^2/m)^{-1/2}. \tag{12}$$

As before one must require $\delta < d < R/n$. This gives $(4\pi R^2 d)\mathcal{N} \gg \frac{1}{2}mn^2d/e^2$ which is just a stronger version of the lower bound on $\mathcal{N}$ we got before. Repeating the previous discussion *verbatim* shows that $2\pi RE \gg n^4$, which bounds Page's $\ln(3n)$ entropy confortably.

In conclusion, bound (1) is satisfied by the system photons + charge carriers. It should be clear that the entropy of the conducting material with its many carriers, which we have been ignoring, may well dominate that in photons. Here one can fall back on the usual argument [1] that in a random assembly of particles the entropy is of order of the number of particles, and that each particle's Compton length is necessarily smaller than the system's radius. From these two conditions it follows that the entropy bound applies—with room to spare—to the charge carrier system by itself. Putting all this together makes it clear that it applies to the complete onion system as well.

## IV. COAXIAL CABLE LOOP

Page's third example [12] is furnished by an electromagnetic field confined to a coaxial cable of length $L$ which is coiled up so as to fit within a sphere of radius $R$, with $R \ll L$, before being connected end to end to form a closed loop. Page's entirely qualitative reasoning proceeds by analogy with a rectilinear coaxial cable with periodic boundary conditions. A rectilinear infinitely long coaxial cable has some electromagnetic modes which propagate along its axis with arbitrarily low frequency. Page notes that for the coiled-up cable, each right moving mode is accompanied by a degenerate (in frequency) left moving mode (basically this follows from time reversal invariance of Maxwell's equations). He then argues that if the cable's *outer* radius $\varrho$ is thin on scale $R$, the structure of the electromagnetic modes is little affected by the cable's curvature. This leads him to estimate the lowest frequency $\omega_1$ as similar to that of the rectilinear coaxial cable with periodic boundary conditions with period $L$: $\omega_1 \approx 2\pi L^{-1}$. Page then imagines a mixed electromagnetic state of energy $E - E_{\text{vac}} = 2\pi L^{-1}$ in which a single photon occupies the right- or the left-moving mode of frequency $\omega_1$ with equal probabilities $\frac{1}{2}$. The entropy of this state is $\ln 2$. However, $2\pi(E - E_{\text{vac}})R \approx 4\pi^2(R/L)$ which could be very small compared to $\ln 2$. Therefore, Page exhibits this example as a violation of the Schiffer-Bekenstein 'strong entropy bound'.

As I stressed in Sec.I, the strong bound was never formalized in a theorem for electromagnetism. If Page is correct in his estimate of $\omega_1$, such a theorem cannot apply to a cavity with not simply connected crossection, like the coaxial cavity. (However, the myriad examples studied numerically [5,6] strongly suggest that a theorem of the desired type should exist for simply connected cavities.) At any rate, in the present example the interesting question is whether the coaxial cavity plus electromagnetic field obeys the original entropy bound (1).

The inner conductor of the cable—let its outer radius be $\rho$ and its thickness $d$—is an essential part of the system, for without it the lowest propagating frequency would be $O(\varrho^{-1})$, i.e., very large on scale $L^{-1}$. Now in order for the inner conductor to keep the fields out of it, as required by the whole notion of a coaxial cable, one must have either $\rho > d > \omega_p^{-1}$ or $\rho > d > \delta$ (see Sec. III).

In the case $\rho > d > \omega_p^{-1}$, Eq. (11) informs us that $\mathcal{N}\rho d > m(4\pi e^2)^{-1}$ where, as in Sec. III, $m$ denotes a charge carrier's mass and $\mathcal{N}$ the carrier density. Since the volume of material in the inner conductor is $\pi L[\rho^2 - (\rho - d)^2]$, its mass energy is at least $\pi m L \mathcal{N}\rho d$, and thus the total mass-energy $E$ of cable plus field is constrained by $E > m^2 L(4e^2)^{-1}$. But a charge carrier has to be localized within the conductor, which requires that $md > m\rho \gg 1$. Hence $E \gg L(4e^2\rho^2)^{-1}$ so that $2\pi ER \gg (\pi/2e^2)(L/\rho)(R/\rho)$. Now obviously $R > \varrho > \rho$ and $L \gg R$ by the conditions of the problem, while $e^2 < 1$ by the condition of stability (see Sec. III), so that $2\pi ER \gg 1$. Thus the bound on entropy confortably bounds Page's entropy $\ln 2$.

Were the mixed state instead to involve one photon in any one of the low lying modes having no crossectional nodes and wavelength along the cable of the form $k(L/2\pi)$ with $k$ an integer, one could get a bigger entropy. There are $O(L/\rho)$ such modes with frequency below that of the lowest lying transversely excited mode (which is obviously of order $\rho^{-1}$), so the entropy of the envisaged state is $\sim \ln(L/\rho)$. Because $\ln x < x$ for $x > 1$, this is obviously bounded by $(L/\rho)(R/\rho)$ and, therefore, by $2\pi ER$.

In the case $\rho > d > \delta$, Eq. (12) gives $\mathcal{N}\rho d \gg m(2\pi e^2)^{-1}$. This is just a stronger version of the earlier lower bound on $\mathcal{N}$. Repeating the previous discussion shows again that $2\pi ER$, with $E$ the total energy of the system, bounds the entropy.

## V. LOW TEMPERATURE SYSTEMS

Page also examined [11] the generic example in which the system's density matrix is diagonal and involves $g+1$ equally probable pure states. The entropy is $\ln(g+1)$. Obviously the greatest challenge to bound (1) is posed when $g$ of the states are degenerate and just a small energy $\Delta$ above the (unique) ground state whose energy is $\epsilon_0$. Now for any $\Delta$ and $g$ the mean energy $E$ of the complete system is *at least* $\epsilon_0$, so $2\pi RE > 2\pi R\epsilon_0$. The present system is so generic that it is not feasible to estimate $\epsilon_0$ as I did previously. It should be clear, however, that the system's longest possible Compton wavelength, $\epsilon_0^{-1}$, should lie well below $R$; otherwise the contention that the system fits within a definite radius $R$ would make no sense as it would be poorly localized on scale $R$. It is probably conservative to take $R\epsilon_0 > 3$. Therefore, no violation of the entropy bound can occur for $g < 10^8$. Can $g$ be bigger ?

Now quantum mechanical systems have low degeneracies in the low lying levels. Quantum field systems have more. But even in those systems with accidental degeneracies, e.g. electromagnetic field in a cubical box, $g < 10$ [5]. As the end of Sec.II B illustrates, it is hard to generate high degeneracies in nonlinear fields in a box. One could create a highly degenerate state artificially by lumping together states closely spaced in energy, e.g. the one-photon states using the modes with $k = 1, 2, \cdots$ of the coaxial cable. But as shown in Sec.IV, the large $L/\rho$ required to have many of these modes defeats the attempt to keep $2\pi RE$ small because a long cable involves a lot of charge carriers. This illustrates the point

that, unlike Page, one cannot arbitrarily legislate a high degeneracy. Rather, one must examine the energy cost paid by the passive components of the system in providing a large number of closely spaced states for the active part which can be consolidated into a formally highly degenerate one.

In a related line of thought, Page [11] revives an old challenge to the entropy bound [17], which is occasionally discovered anew [19,20]: a system in a thermal state seems to violate the entropy bound if its *inverse* temperature $\beta$ is sufficiently high. The essence and resolution of the problem is captured by the following purely analytical treatment. Consider a system of radius $R$ with ground energy $\epsilon_0$, a $g$-fold degenerate excited state at energy $\epsilon_1 = \epsilon_0 + \Delta$, and higher energy states. For sufficiently large $\beta$ one may neglect the higher energy states in the partition function $Z = \sum_i \exp(-\beta \epsilon_i)$, and so approximate it by $\ln Z \approx -\beta \epsilon_0 + \ln(1 + g e^{-\beta \Delta})$. The *mean* energy is

$$E = -\frac{\partial \ln Z}{\partial \beta} = \epsilon_0 + \frac{g\Delta}{e^{\beta \Delta} + g} \tag{13}$$

while the entropy takes the form

$$S = \beta E + \ln Z = \frac{g\beta\Delta}{e^{\beta \Delta} + g} + \ln(1 + g e^{-\beta \Delta}). \tag{14}$$

The typical claim [11,17] is "measure energies from the ground state so that $\epsilon_0 = 0$; then for $\beta > 2\pi R$ one gets $S > 2\pi R E$ and the bound is violated".

But we have already seen in Sec. II that taking the zero of energy of a system at its ground state is not automatically justified because it may mean using as $E$ something distinct from the gravitating energy. And I have already discussed why $R\epsilon_0$ should be at least of the order a few, say 3.

The interesting quantity now is

$$S - 2\pi R E = \Xi(\beta\Delta) \equiv \frac{g(\beta\Delta - 2\pi R\Delta)}{e^{\beta\Delta} + g} + \ln(1 + g e^{-\beta\Delta}) - 2\pi R\epsilon_0. \tag{15}$$

The function $\Xi(y)$ has a single *maximum* at $y = 2\pi R\Delta$ where $\Xi = \ln(1 + g e^{-2\pi R\Delta})$. I thus conclude that

$$S < 2\pi R E + [\ln(1 + g e^{-2\pi R\Delta}) - 2\pi R\epsilon_0]. \tag{16}$$

For the quantity in square brackets to be nonnegative it would be necessary for $g \geq e^{2\pi R\Delta}[e^{2\pi R\epsilon_0} - 1]$, i.e., $g > 10^8$. As we saw above, this cannot be arranged. Thus the quantity in square brackets in Eq. (16) has to be negative: *for sufficiently low temperature the entropy bound is upheld* with room to spare. The above argument also illustrates how to deal with any mixed state which ascribes low probabilities to the high energy pure states.

Early realistic numerical calculations of thermal quantum fields in boxes [15] did reveal that, were the ground state energy to be ignored, the bound on entropy would be violated at very low temperatures, typically when $E < 10^{-9}R^{-1}$ ($R$ enters through the "energy gap" $\Delta$). It was also clear early [1,15] that taking any reasonable ground state energy into account precludes the violation. As the temperature rises, more and more pure states are excited, and eventually $S/E$ peaks and begins to decrease. In this regime the entropy bound is always obeyed regardless of whether or not one includes $\epsilon_0$ in the total energy [5].

## VI. PROLIFERATION OF SPECIES

Page also revives the old "proliferation of species" challenge to the entropy bound [18,19]. Suppose there were to exist as many copies $N$ of a field e.g. the electromagnetic one, as one ordered. It seems as if the entropy in a box containing a fixed energy allocated to the said fields should grow with $N$ because the bigger $N$ is, the more ways there are to split up the energy. Thus eventually the entropy should surpass the entropy bound. Numerical estimates show that it would take $N \sim 10^9$ to do the trick [15]. A similar picture seems to come from Eq. (16); the degeneracy factor $g$ should scale proportionally to $N$ making the factor in square brackets large, so that, it would seem, one could not use the argument based on (16) to establish that $S < 2\pi RE$. However, as recognized in refs. [2,3,14], the above reasoning fails to take into account that each field species makes a contribution of zero point fluctuations energy which gets lumped in $\epsilon_0$. If these contributions are positive, then the negative term in the square bracket eventually dominates the logarithm as $N$ grows, and for large $N$ one again recovers the entropy bound [2]. If they are negative (which implies a Casimir suction proportional to $N$ on the walls which delineate the system), then the scalar field example suggests that the wall energy, which must properly be included in $\epsilon_0$, should suffice to make the overall $\epsilon_0$ positive [2]. Again the entropy bound seems safe.

There is an alternative view [27,3]: the seeming clash between entropy bound and a large number of species merely tells us that physics is consistent only in a world with a limited number of species, such as the one that is observed. Indeed, as Brustein, Eichler and Foffa [21] have argued, a large number of field species will make the vacuum of quantum field theory unstable against collapse into a "black hole slush". Thus the proliferation of species argument against the entropy bound is not even physically consistent.

## VII. CAVEATS

I have stressed the robustness of the entropy bound for weakly gravitating systems. But one should recall that the bound has its limitations. These principally belong to the strongly gravitating system regime. Bound (1) does not apply in wildly dynamic situations such as those found inside black holes [28], and it is not guaranteed to work for large pieces of the universe (which, after all, are not complete systems).

## ACKNOWLEDGMENTS

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